SOLUTIONS TO EXAM 1, MATH 10550

1. *Find points where the function* $f(x) = \dfrac{x^2-1}{x^3-4x}$ *is not continuous?*

Solution: Factor the denominator of the function to get $x(x-2)(x+2)$. The function will be discontinuous when this is equal to 0 which is exactly when $x = 0$ and $x = \pm 2$.

2. *Compute*

$$\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x-2}.$$

Solution: Directly plugging 2 into the given function gives 0 in the denominator and 0 in the numerator. Therefore, we need to rationalize by multiplying the numerator and denominator by $\sqrt{x^2+5}+3$. Following this through gives us the following:

$$\begin{aligned}
\lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x-2} \cdot \frac{\sqrt{x^2+5}+3}{\sqrt{x^2+5}+3} &= \lim_{x\to 2} \frac{x^2+5-9}{(x-2)\sqrt{x^2+5}+3} \\
&= \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)\sqrt{x^2+5}+3} \\
&= \lim_{x\to 2} \frac{x+2}{\sqrt{x^2+5}+3} \\
&= \frac{4}{6} = \frac{2}{3}
\end{aligned}$$

3. *Find* $f'(4)$ *if*

$$f(x) = 4\sqrt{x} - \frac{16}{\sqrt{x}}.$$

Solution: First, calculate $f'(x)$.

$$f'(x) = 4(\frac{1}{2})x^{-\frac{1}{2}} - 16(-\frac{1}{2})x^{-3/2} = 2x^{-\frac{1}{2}} + 8x^{-3/2}$$

So $f'(4) = \frac{2}{\sqrt{4}} + \frac{8}{4^{3/2}} = 1 + 1 = 2.$

4. *Compute the derivative of*

$$f(x) = \frac{x+\cos x}{x+\sin x}.$$

Solution: Recalling the quotient rule, with $h(x) = x + \cos x$, $g(x) = x + \sin x$, so $f(x) = \frac{h(x)}{g(x)}$, $h'(x) = 1 - \sin x$, $g'(x) = 1 + \cos x$ So

$$f'(x) = \frac{h'(x)g(x) - g'(x)h(x)}{g(x)^2} = \frac{(1-\sin x)(x+\sin x) - (1+\cos x)(x+\cos x)}{(x+\sin x)^2}$$

5. *Find all the horizontal tangent lines to the curve* $y = \dfrac{1}{1+x^2}$.

Solution: First, note that

$$y = \frac{1}{1+x^2} = (1+x^2)^{-1}.$$

Now differentiate the function using the chain rule:

$$y' = (-1)(1+x^2)^{-2}(2x) = \frac{-2x}{(1+x^2)^2}.$$

For each x, y' gives the slope of the tangent line to the curve at x. Since horizontal lines have slope 0, we must find when $y' = 0$. This happens only when $-2x = 0$, that is, when $x = 0$. Therefore, the only horizontal tangent line to the curve occurs at $x = 0$. Since

$$y(0) = \frac{1}{1+(0)^2} = 1,$$

the horizontal tangent line is $y = 1$.

6. *Find the derivative of* $f(x) = (1+\sin(x^2))^{1/4}$.

Solution: We can write $f(x)$ as the composition of three functions: $g(x) = x^{\frac{1}{4}}, h(x) = 1+\sin(x)$, and $p(x) = x^2$. We have $f(x) = g(h(p(x)))$. Thus, using the chain rule twice gives:

$$\begin{aligned} f'(x) &= g'(h(p(x)))(h'(p(x))p'(x)) \\ &= \frac{1}{4}(1+\sin(x^2))^{-\frac{3}{4}}(\cos(x^2)(2x)) \\ &= \frac{1}{2}x(1+\sin(x^2))^{-\frac{3}{4}}\cos(x^2). \end{aligned}$$

7. *If* $f(x) = x^2\cos x + \sin x$, *find* $f''(x)$.

Solution: $f'(x) = \cos x - x^2\sin x + 2x\cos x$, $f''(x) = 2\cos x - \sin x - x^2\cos x - 4x\sin x$.

8. *Compute*

$$\lim_{x\to 0}\frac{1-\cos x}{x\tan x}.$$

Solution:

$$\begin{aligned} \lim_{x\to 0}\frac{1-\cos x}{x\tan x} &= \lim_{x\to 0}\frac{(1-\cos x)}{x\tan x}\frac{(1+\cos x)}{(1+\cos x)} \\ &= \lim_{x\to 0}\frac{\sin^2 x}{x\tan x(1+\cos x)} \\ &= \lim_{x\to 0}\frac{\sin x\cos x}{x(1+\cos x)} \\ &= \lim_{x\to 0}\frac{\sin x}{x}\times\lim_{x\to 0}\frac{\cos x}{1+\cos x} \\ &= \frac{1}{2}. \end{aligned}$$

9. *If* $f'(2) = 5$, $g(4) = 2$, $g(2) = 1$, $f(2) = -1$ *and* $g'(4) = 3$, *find* $(f\circ g)'(4)$.

Solution: Using the chain rule, we have

$$\begin{aligned}(f\circ g)(4) &= f'(g(4))g'(4)\\ &= f'(2)g'(4)\\ &= 15.\end{aligned}$$

10. *If* $\sin(\pi xy) = \pi(x+y)$ *find* $\frac{dy}{dx}$ *at* $(1,-1)$ *by implicit differentiation.*
Solution: Differentiating both sides with respect to x,

$$\cos(\pi xy)\left(\pi y + \pi x\frac{dy}{dx}\right) = \pi\left(1+\frac{dy}{dx}\right).$$

Now, letting $x = 1$, $y = -1$ we get

$$\pi\cos(-\pi)\left(-1+\frac{dy}{dx}\right) = \pi(1+\frac{dy}{dx}).$$

Isolating dy/dx we get

$$\frac{dy}{dx} = 0.$$

11. *Find the derivative of*

$$y = \frac{1}{1-x}$$

using the definition of the derivative.
Solution: Let $f(x) = \frac{1}{1-x}$. Then the definition of derivative says

$$\begin{aligned}f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\\ &= \lim_{h\to 0}\frac{\frac{1}{1-(x+h)}-\frac{1}{1-x}}{h}\\ &= \lim_{h\to 0}\frac{\frac{1-x}{(1-x-h)(1-x)}-\frac{1-x-h}{(1-x-h)(1-x)}}{h}\\ &= \lim_{h\to 0}\frac{\frac{(1-x)-(1-x-h)}{(1-x-h)(1-x)}}{h}\\ &= \lim_{h\to 0}\frac{\frac{h}{(1-x-h)(1-x)}}{h}\\ &= \lim_{h\to 0}\frac{1}{(1-x-h)(1-x)}\\ &= \frac{1}{(1-x)(1-x)} = \frac{1}{(1-x)^2}\end{aligned}$$

12. *For what points* P *and* Q *on the graph of the function* $y = x^2$ *does the tangent line at that point pass through the point* $(0,-1)$*?*
Hint: Write down the equation for the tangent line through the point (a, a^2) and proceed from there.
Solution: First, $y = f(x) = x^2$, and so $f'(x) = 2x$. Recall that the slope of each these tangent lines at P and Q is the derivative of f at the x coordinate of P and at Q. Using the hint, we see that $f'(a) = 2a$. By the picture, note that the tangents at

P and Q are lines that have the same y intercept, -1. So the equation for each of the lines is $y = 2ax - 1$ for two different values of a. Using the hint, with $(x, y) = (a, a^2)$ in the equation of the tangent line $y = 2ax - 1$, we get $a^2 = 2a^2 - 1$, or $a^2 = 1$, which (conveniently) has two possible solutions: $a = \pm 1$. Plugging ± 1 into our original equation $y = x^2$ we get $P = (1, 1)$, $Q = (-1, 1)$.

13. *Show that there is at least one solution of the equation*

$$x^3 = 3x^2 - 1.$$

Justify your answer, identify the theorem you use and explain why the theorem applies.

Solution: We can rewrite this equation as $x^3 - 3x^2 + 1 = 0$. Set $f(x) = x^3 - 3x^2 + 1$. There is at least one solution to our original equation if $f(x)$ is zero for some x. Note that $f(0) = 1$ and $f(1) = -1$, so 0 is between $f(0)$ and $f(1)$. The function $f(x)$ is a polynomial, so it is defined and continuous on the entire real line and in particular on the interval $[0, 1]$. Therefore, the intermediate value theorem applies, and tells us for some real number a in the interval $(0, 1)$, $f(a) = 0$. Thus, $0 = f(a) = a^3 - 3a^2 + 1$, so $a^3 = 3a^2 - 1$, and a is a solution to the original equation.